Isomonodromic deformations of the sl(2) Fuchsian systems on the Riemann sphere

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Abstract

This paper is devoted to two geometric constructions related to the isomonodromic method. We follow Drinfelds ideas from [D3] and develop them in the case of the curve $X = \mathbb{P}^1 \backslash \{a_1, \ldots, a_n\}$. Thus we generalize the results of [AL] to the case of arbitrary number n of points. First, we construct separated Darboux coordinated in terms of the Hecke correspondences between moduli spaces. In this way we present a geometric interpretation of the Sklyanin formulas from [Skl]. In the second part of the paper, we construct Drinfeld's compactification of the initial data space and describe the compactifying divisor in terms of certain FH-sheaves. Finally, we give a geometric presentation of the dynamics of the isomonodromic system in terms of deformations of the compactifying divisor and explain the role of apparent singularities for Fuchsian equations. To illustrate the results and methods, we give an example of the simplest isomonodromic system with four marked points known as the Painleve-VI system.

1 Introduction

In this work we present a geometric interpretation of the isomonodromic deformation of a non-resonant $sl(2)$ Fuchsian system on the Riemann sphere; this isomonodromic system is known as the Schlesinger system. Given a generic Fuchsian differential equation of order N with singularities at $S := \{a_1, \ldots, a_n\}$ on $\mathbb{P}^1$ and let us put it into an isomonodromic analytical family in the following way.

Consider the Fuchsian system of differential equations $\dfrac{d}{dz} Y(z) = \left(\sum_{i=1}^n \dfrac{B_i}{z - a_i} \right) Y(z)$ with matrix coefficients $B_i \in Mat(N, \mathbb{C})$. Note from the very beginning that we consider only non-resonant systems, that is we assume that $\lambda_i^{(a)} - \lambda_i^{(b)} \notin \mathbb{N}$ for the eigenvalues

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$\{\lambda_i^{(a)}\}$ of matrices B_i, $i = 1, \ldots, n$. Let $Y(z)$ be the fundamental solution of this system. Then consider the equation $\partial_z Y(z) = L(z)Y(z)$, where

$$L(z) = \sum_{i=1}^{n} \frac{B_i(a_1, \ldots, a_n)}{z - a_i} dz$$

simultaneously with the following (isomonodromic) condition for the coefficients $B_i(a_1, \ldots, a_n)$, $i = 1, \ldots, n$

$$dB_i(a_1, \ldots, a_n) = \sum_{j=1}^{n} [B_j, B_i] \, d\log(a_i - a_j)$$

called the Schlesinger equation. The last equation indicates the complete integrability condition $d\omega = \omega \wedge \omega$ for the matrix-valued 1-form $\omega = dY(z) \cdot Y(z)^{-1}$; in other words, it is the zero-curvature condition for the logarithmic connection $\nabla := d - \omega$ in a trivial rank N bundle on the configuration space $\mathbb{P}^1 \times \mathbb{C}^n$. Such systems were investigated originally by Schlesinger (see [Sch]) and later algebraic aspects were considered by Flashka and Newell (see [FN]), Jimbo and Miwa (see [JM]). Geometric aspects of the isomonodromic systems were originally initiated by Röhrl in [R], and then developed by Bolibruch (see [B], [AB]), Hitchin (see [Hit]), Arinkin and Lysenko ([AL]) in various senses. In our approach we develop Drinfeld's ideas (see [D2], [D3]) to study the isomonodromy problem from the point of view of geometric representation theory; in particular, we generalize the results of [AL] to the case of arbitrary number of singularities.

In this paper we describe the fundamental matrix of our Fuchsian system of rank 2 in terms of horizontal sections of a certain rank 2 bundle $\mathcal{L}$ with respect to the logarithmic $sl(2)$-connection on the Riemann sphere $\mathbb{P}^1$

$$\nabla : \mathcal{L} \longrightarrow \mathcal{L} \otimes \Omega_{\mathbb{P}^1}^1(a_1 + \ldots + a_n)$$

with $\mathrm{Res}_{a_i} \nabla = B_i$.

Consider the co-adjoint representation of the group $G = SL(2) \to \mathrm{End}(sl(2)^*)$, $X \mapsto \mathrm{ad}_X^*$; herewith we assume that the coefficients B_i lie in co-adjoint $sl(2)$-orbits $\mathcal{O}_i$. Fixing the eigenvalues of the residues B_i we fix the appropriate $sl(2)$-orbits. Every $sl(2, \mathbb{C})$-orbit is a 2-dimensional non-compact variety with a natural symplectic form which in the co-adjoint representation is $\omega_\xi(X, Y) = -\langle \mathrm{ad}_X^*, Y \rangle$ for any $\xi \in sl(2)^*$. The symplectic quotient of the direct product of $SL(2, \mathbb{C})$-orbits is a symplectic variety of the following dimension

$$\dim \Big(\prod_{i=1}^{n} \mathcal{O}_i // SL(2, \mathbb{C}) \Big) = n \cdot \dim \mathcal{O}_i - 2 \cdot \dim SL(2, \mathbb{C}) = 2(n - 3).$$

Identify the symplectic quotient with the initial data space and present it as an open subset of the coarse moduli space $\mathcal{M}_n(2)$ of collections

$$(\mathcal{L}, \nabla; \quad \phi : \mathrm{Det}\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}; \quad \lambda_1, \ldots, \lambda_n),$$

of a rank 2 bundle $\mathcal{L}$ on $\mathbb{P}^1$ with fixed determinant and a connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega_{\mathbb{P}^1}^1(a_1 + \ldots + a_n)$ such that the eigenvalues of the residues of the connection $\mathrm{Res}_{a_i} \nabla = B_i$

at a_i, $i = 1, \ldots, n$ are $\{\lambda_i, -\lambda_i\}$. Note that in this work we consider only the coarse moduli spaces and assume this even when the word "coarse" is omitted.

In his remarkable thesis [D2] Drinfeld introduced original geometric objects; they are elliptic modules and the Frobenius-Hecke sheaves. In our paper we follow the geometric ideas of Drinfeld and apply them to investigate the isomonodromic deformation of the Fuschsian systems of rank two. We pay a special attention to the bounds of application for the procedure of separation of variables. In particular, we construct Drinfeld's compactification (see [D2]) of the initial data space of the system and investigate the cases in which the classical procedure of separation of variables does not work. Also emphasize that our construction naturally entails the identification of the phase space of a Fuchsian system of differential equations (initial data space) with the phase space of the isomonodromic deformation of the system. In this way one may understand this paper as a geometric presentation of the *isomonodromic method* of investigation of Fuchsian differential equations.

The paper is organized in the following way. In the first part (Section 3) we construct geometric Darboux coordinates on $\mathcal{M}_n(2)$ and notice that the result coincides with the calculations ("magic recipe") from [Skl].

Let us point that we omit the assumption of the triviality of the bundle $\mathcal{L}$ though fixing its determinant by a horizontal isomorphism $\phi : \mathrm{Det}\mathcal{L} \simeq \mathcal{O}$. We consider the moduli space $\mathcal{M}_n(2)$ of pairs $(\mathcal{L}, \nabla)$ equipped with ϕ and with fixed eigenvalues of the residues of the connection. We construct a parametrization of the moduli space in the sense of Drinfeld and in this way we give a geometric interpretation of the Sklyanin's formulas from [Skl]. For this purposes we have to impose a notion of *stability* for our configurations. We discuss it and investigate the isomonodromic system for the (semi)stable configurations. Note, that addition of the strata of $\mathcal{M}_n(2)$ which correspond to non-trivial bundles allows to uncover the hidden symmetries of the system; in particular, the discrete symmetries of the isomonodromic system , calculated in [O], can be explained only in terms of the completed initial data space $\mathcal{M}_n(2)$ (see the example in the end of this paper).

The second part of the paper (Section 4) contains the construction of Drinfeld's compactification of the initial data space $\mathcal{M}_n(2)$. Below we present two naive recipes how to complete the initial data space of the isomonodromic deformation. They are very simple and explicit; however, they are both generalized by our construction of the compactification and illustrate it.

Consider the cotangent bundle Ω on $\mathbb{P}^1$ and denote by $\mathrm{Tot}(\mathbb{P}^1, \Omega(\sum a_i))$ the total space of the bundle $\Omega(a_1 + \ldots + a_n)$. Our construction of the Darboux coordinates provides the description of the initial data space $\mathcal{M}_n(2)$ in terms of the $(n-3)$-th symmetric power of the non-compact surface $K_n := \mathrm{Tot}(\mathbb{P}^1, \Omega(\sum a_i))$ (see for example [GNR]). Precisely, consider the compact surface $\overline{K_n} = \mathbb{P}(\mathcal{O} \oplus \Omega(\sum a_i)) = s_\infty \sqcup K_n$ for s_∞ the infinite section s_∞ and let $F_i := \Omega(\sum a_i)|_{a_i} \subset \overline{K_n}$. The fibres F_i, $i = 1, \ldots, n$ are trivialized by the residue map $R : F_i \widetilde{\to} \mathbb{C}$. Let us blow-up the surface $\overline{K_n}$ at $2n$ points $R^{-1}(\lambda_i^{\pm}) \in F_i$ for $\{\lambda_i^{\pm}\} = \{\lambda_1, 1 - \lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_n, -\lambda_n\}$ and consider the non-compact surface

$$K_n' := (\mathrm{Bl}_{R^{-1}(\pm\lambda_i)}\overline{K_n}) \setminus (s_\infty \cup \widetilde{F_1} \cup \ldots \cup \widetilde{F_n}),$$

where $\widetilde{F_i}$ are the proper pre-images of the fibers F_i, $i = 1, \ldots, n$. There is a map $\mathcal{M}_n(2) \longrightarrow (K_n')^{(n-3)} := (K_n')^{n-3}/\mathfrak{S}_{n-3}$ that is an isomorphism at the generic point and we thoroughly describe points where it is not an isomorphism.

There is no an ordering on the set of the variables $\{x_i, p_i\}$, $i = 1, \ldots, n-3$ and hence we have to consider either the quotient $(K_n')^{(n-3)} := (K_n')^{n-3}/\mathfrak{S}_{n-3}$ or the $(n-3)!$-covering

$$\widetilde{\mathcal{M}_n(2)} \simeq (K_n')^{n-3}.$$

On the covering $\widetilde{\mathcal{M}_n(2)}$ we have a natural symplectic form

$$\varpi = \sum_{i=1}^{n-3} dx_i \wedge dp_i$$

and it equips $\widetilde{\mathcal{M}_n(2)}$ with a structure of symplectic variety. It is natural to complete $\mathcal{M}_n(2)$ with a pole-divisor of the symplectic form ϖ. We present the compactifying divisor as the pole-divisor of ϖ in 5.2.

Another natural way to regard the compactification problem is as follows. Consider an algebraic curve C on K_n defined by the equation $R(z, \lambda) = 0$ for

$$R(z, \lambda) := (\det L(z) - \lambda \cdot Id).$$

It is known as *spectral curve*; the genus of C is $n - 3$, which is equal to the half of dimension of the initial data space. Remarkably, C is not preserved by the isomonodromic deformation and this fact entails the natural completion of the phase space with a limit cycle of the spectral curve C; this construction of completion of the initial data space is a part of the isomonodromic method.

It is significant that it is possible to perform a natural compactification of the initial data space $\mathcal{M}_n(2)$ in terms of a degenerated model of the curve C. Consider the surface $\overline{K_n}$, trivialized fibers F_i, $i = 1, \ldots, n$ and the infinite section s_∞ on it; let $\{F_i, s_\infty\}$ be the basis in the homology group $H_2(\overline{K_n}, \mathbb{Z})$. The intersection numbers are

$$F_i \cdot F_j = 0, \quad s_\infty \cdot s_\infty = -\deg \Omega(a_1 + \ldots + a_n) = 2 - n, \quad F_i \cdot s_\infty = 1, \quad C \cdot F_i = 2;$$

besides, the intersection number of the curve C with s_∞ is zero. The topological class of C is preserved by the isomonodromy deformation. In this way we compactify the initial data space $\mathcal{M}_n(2)$ with the divisor D such that its factors $\Theta_{(i)} \subset \overline{K_n'}$ preserve the topological invariant and $\Theta_{(i)} \cdot s_\infty = 0$; for $n = 4$ this immediately implies $\Theta = 2s_\infty + F_1 + \ldots + F_4$. In the case $n > 4$ the above argument is not so explicit and we obtain the same result in 4.2 using the FH-sheaves approach to the compactification problem.

Besides, in the fourth section we emphasize the important role of the complete self-intersection of the compactifying divisor: $\boldsymbol{\Theta}_n := D \cdot D$ whose dimension is exactly $n-3$. We describe the dynamics of the isomonodromic system in terms of the cycle $\boldsymbol{\Theta}_n$. Finally we explain the role of the apparent singularities of the Fuchsian systems originally introduced in [F](see also [B] and [AB]). Precisely, we identify the cycle $\boldsymbol{\Theta}_n$ with the moduli space of the collections

$$(\widetilde{\mathcal{L}_{\Theta_n}}, \nabla_{\Theta_n}; \phi' : \operatorname{Det} \widetilde{\mathcal{L}_{\Theta_n}} \widetilde{\to} \mathcal{O}(-a_1 - \ldots - a_{n-2}); (\widetilde{\lambda_i^+}, \widetilde{\lambda_i^-}), i = 1, \ldots, n)$$

for some $a \in S$, where $\widetilde{\mathcal{L}_{\Theta_n}}$ is the rank 2 bundle of degree $2 - n$ with the logarithmic connection ∇_{Θ_n} such that the eigenvalues of $\operatorname{Res}_{a_i} \nabla_{\Theta_n}$ are $(\widetilde{\lambda_1^+}, \widetilde{\lambda_1^-}) := (\lambda_i, 1 - \lambda_i)$ at

a_i, $i = 1, \ldots, n-2$ and $(\widetilde{\lambda_i^+}, \widetilde{\lambda_i^-}) := (\lambda_i, -\lambda_i)$ at $a_i = a_{n-1}$, a_n. We present the dynamical variables $\{x_i, p_i\}$, $i = 1, \ldots, n-3$ of the isomonodromic deformation as the parameters of the Hecke correspondence between $\mathbf{\Theta}_n$ and the moduli space $\mathcal{M}'_n(2) \simeq \mathcal{M}_n(2)$ of the collections

$$(\widetilde{\mathcal{L}}, \widetilde{\nabla} := \nabla|_{\widetilde{\mathcal{L}}}; \widetilde{\phi} : \mathrm{Det}\,\widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1); (\lambda_1^+, \lambda_1^-), \ldots, (\lambda_n^+, \lambda_n^-)),$$

where $\widetilde{\mathcal{L}}$ is a rank 2 bundle on $\mathbb{P}^1$ with fixed horizontal isomorphism $\widetilde{\phi} : \mathrm{Det}\mathcal{L} \simeq \mathcal{O}(-a_1)$ and with a connection $\widetilde{\nabla}$ with singularities at $\{a_1, \ldots, a_n\}$; the eigenvalues of $\mathrm{Res}_{a_i} \widetilde{\nabla}$ are $(\lambda_1^+, \lambda_1^-) := (\lambda_1, 1 - \lambda_1)$ at a_1 and $(\lambda_i^+, \lambda_i^-) := (\lambda_i, -\lambda_i)$ at a_i, $i = 2, \ldots, n$. In terms of the connections

$$\widetilde{\nabla} = \nabla_{\Theta_n}(p_1, \ldots, p_{n-3}) - \sum_{i=1}^{n-3} \mathbf{P}_{p_i} \frac{dz}{z - x_i},$$

where $\mathbf{P}_{p_i}$ are the projectors on the invariant one-dimensional subspaces $p_i \subset \widetilde{\mathcal{L}_{\Theta_n}}|_{x_i}$, $i = 1, \ldots, n-3$. The terms $\mathbf{P}_{p_i} \dfrac{dz}{z - x_i}$ do not change the monodromy of the connection and the points $x_1, \ldots, x_{n-3}$ are called *the apparent singularities* of the connection $\widetilde{\nabla}$.

1.1 Acknowledgements

2 Modificaitons of logarithmic sl(2)-connections

In [D1] Drinfeld presented a construction of elliptic module which generalized a set of classical algebraic ideas; then in [D2] the Frobenius-Hecke sheaves, (or, "shtukas") were introduced. These new concepts provided a new understanding of the Langlands conjecture for automorphic forms, and led to establishing this conjecture in the case $GL(2)$ over function field. Besides, this approach uncovered profound relations between arithmetic and algebraic geometry, representation theory and differential equations.

For our purposes it will be convenient to modify the original definition from [D2] and to introduce the following.

Definition. A Frobenius-Hecke sheaf (FH-sheaf) of level K (for an integer K) on $\mathbb{P}^1$ is a flag of locally free sheaves of the same rank $\mathcal{F}_0 \subset \mathcal{F}$ on $\mathbb{P}^1$ such that the codimension of the support $\mathrm{supp}\,(\mathcal{F}/\mathcal{F}_0) \subset \mathbb{P}^1$ equals one and $(\mathcal{F}/\mathcal{F}_0)$ has a K-dimensional space of sections. For a generic FH-sheaf all the points of $\mathrm{supp}\,(\mathcal{F}/\mathcal{F}_0)$ are distinct that is $\mathcal{F}/\mathcal{F}_0$ is isomorphic to a sum of sky-scraper sheaves $\bigoplus \delta_{x_i}$ and each sky-scraper sheaf δ_{x_i} has a one-dimensional space of sections.

Between the moduli spaces of FH-sheaves $(\mathcal{F}'_1 \subset \mathcal{F}_1)$ and $(\mathcal{F}'_2 \subset \mathcal{F}_2)$ of different levels K_1

and K_2 there are correspondences, called the Hecke correspondences. These correspondences are performed by modifications (see [D3]) of the locally free sheaves $\mathcal{F}_i'$, $\mathcal{F}_i$; upper modifications reduce the level and lower ones increase it.

Given a rank 2 bundle $\mathcal{L}$ on $\mathbb{P}^1$ with a connection ∇, let $x \in \mathbb{P}^1$. Denote by V a fiber $\mathcal{L}|_x$ and let $l \subset V$ be a one-dimensional subspace. Identify $\mathcal{L}$ with of its sheaf sections and consider the following locally trivial sheaves.

$$(x, l)^{\mathrm{low}}(\mathcal{L}) := \{s \in \mathcal{L} \mid s(x) \in l\}, \qquad (x, l)^{\mathrm{up}}(\mathcal{L}) := (x, l)^{\mathrm{low}}(\mathcal{L}) \otimes \mathcal{O}(x)$$

which are called the lower and the upper modifications respectively. Denote the lower modification by $\widetilde{\mathcal{L}} := (x, l)^{\mathrm{low}}(\mathcal{L})$ and consider the natural map $\widetilde{\mathcal{L}}|_x \longrightarrow \mathcal{L}|_x$; evidently its image is l. Set $\widetilde{l} := \ker(\widetilde{\mathcal{L}}|_x \longrightarrow \mathcal{L}|_x)$ then $(x, \widetilde{l})^{\mathrm{up}}\widetilde{\mathcal{L}} = \mathcal{L}$. The lower and the upper modifications provide the following exact sequences.

$$0 \longrightarrow (x, l)^{\mathrm{low}}(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow \delta_x \otimes \mathcal{L}_x/l \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{L} \longrightarrow (x, l)^{\mathrm{up}}\mathcal{L} \longrightarrow \delta_x \otimes l \otimes \mathcal{T}_x \longrightarrow 0$$

respectively. Here δ_x is a sky-scraper sheaf with the support at x and $\mathcal{T}_x$ is the localization of the tangent bundle at x.

Roughly speaking, given a local decomposition $V = l \bigoplus \widetilde{l}$ of $\mathcal{L} \simeq V \otimes \mathcal{O}$, we have

$$(x, l)^{\mathrm{low}}(\mathcal{L}) = l \otimes \mathcal{O} \bigoplus \widetilde{l} \otimes \mathcal{O}(-x), \qquad (x, l)^{\mathrm{up}}(\mathcal{L}) = l \otimes \mathcal{O}(x) \bigoplus \widetilde{l} \otimes \mathcal{O}.$$

In other words we change our bundle rescaling the basis of sections in the neighborhood of a point x; if the local basis is $\{s_1(z), s_2(z)\}$ with $l \otimes \mathcal{O} \simeq \langle s_1(z) \rangle$ and $\widetilde{l} \otimes \mathcal{O} \simeq \langle s_2(z) \rangle$ then the basis of the lower modification $(x, l)^{\mathrm{low}}$ of the bundle is generated by the sections $\{s_1(z), (z-x)s_2(z)\}$, and of the upper one $(x, l)^{\mathrm{up}}$ by $\{(z-x)^{-1}s_1(z), s_2(z)\}$. Consequently, in the punctured neighborhood we may represent the action of the modifications by the following gluing matrices.

$$(x, l)^{\mathrm{low}} = \begin{pmatrix} 1 & 0 \\ 0 & (z-x) \end{pmatrix}, \qquad (x, l)^{\mathrm{up}} = \begin{pmatrix} (z-x)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Matrix presentation of the modifications is supposed to be quite obvious, and further on we use it freely. Let us note that in our setting the discussed Hecke correspondences are symplectic (singular) gauge transformations (see [LOZ]).

Now discuss the action of the modifications of an $sl(2)$-connection with logarithmic singularities on the projective line $\mathbb{P}^1$.

Definition. [S] A *modulus* $\mathfrak{M}$ supported at S on an algebraic curve X is a finite set $S = \{a_1, ..., a_n\} \subset X$ equipped with a function assigning a positive integer n_i to every point $a_i \in S$. Sometimes we identify $\mathfrak{M}$ with the effective divisor $\sum n_i \cdot a_i$. In the present work we consider the module

$$\mathfrak{M} = \sum_{i=1}^{n} a_i.$$

Let us look how the modifications change the connection. Suppose we start from some logarithmic (Fuchsian) $sl(2)$-connection ∇ on $\mathcal{L}$ and

$$\nabla : \mathcal{L} \longrightarrow \mathcal{L} \otimes \Omega^1(\mathfrak{M});$$

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this means that ∇ has *simple poles* at the support S of $\mathfrak{M}$. Denote the eigen-subspaces of $\mathrm{Res}_{a_i}\nabla$ by $\ell_i^{\pm} := \ker(\mathrm{Res}_{a_i}\nabla \mp \lambda_i)$ and consider the modifications of our pair $(\mathcal{L}, \nabla)$ in these subspaces. Emphasize that we modify the pairs $(\mathcal{L}, \nabla)$ in $(\mathrm{Res}_x\nabla)$-invariant subspaces of $V \subseteq \mathcal{L}|_x$, otherwise we increase the order of a pole of the connection. Indeed, using the matrix presentation write down the action of the modification of the bundle in a non-invariant subspace at $z = 0$:

$$\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \left[d + \begin{pmatrix} \lambda/z & \varepsilon/z \\ 0 & -\lambda/z \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix} = d + \begin{pmatrix} \lambda/z & \varepsilon/z^2 \\ 0 & -(\lambda+1)/z \end{pmatrix},$$

where z is a local parameter. Here because of the ε in the right upper corner, the second component of the modification is not ∇-invariant.

Besides, note that the lower and upper modifications at any point $x \in \mathbb{P}^1$ change the determinant:

$$\mathrm{Det}(x, l)^{\mathrm{low}}\mathcal{L} = \mathrm{Det}\mathcal{L} \otimes \mathcal{O}(-x), \quad \mathrm{Det}(x, l)^{\mathrm{up}}\mathcal{L} = \mathrm{Det}\mathcal{L} \otimes \mathcal{O}(x).$$

Let us illustrate the techniques that we will use in the next sections. Consider the lower modification $\widetilde{\mathcal{L}}$ with the connection

$$\nabla' : \widetilde{\mathcal{L}} \xrightarrow{\nabla|_{\widetilde{\mathcal{L}}}} \mathcal{L} \otimes \Omega(\mathfrak{M}) \xrightarrow{\mathrm{pr}} \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M})$$

on $\widetilde{\mathcal{L}}$ then on the determinant bundle we get the connection

$$\mathrm{Tr}\nabla' = \mathrm{Tr}\nabla + \frac{dz}{z - x}.$$

Perform a pair of the lower and the upper modifications at points a_i and a_j respectively to get the bundle $\mathcal{L}''$ with the same determinant

$$\mathrm{Det}\mathcal{L}'' = \mathrm{Det}\mathcal{L} \otimes \mathcal{O}(a_j - a_i) \simeq \mathrm{Det}\mathcal{L};$$

to do this we have to fix a set of compatible isomorphisms $\mathcal{O} \simeq \mathcal{O}(a_i - a_j)$ such that

$$\mathcal{O} \simeq \mathcal{O}(a_i - a_j) \otimes \mathcal{O}(a_j - a_k) \simeq \mathcal{O}(a_i - a_k).$$

Nevertheless, if we start from an $sl(2)$-connection ∇, then after such procedure we get the connection

$$\nabla'' = \nabla + \mathbf{P}_{l_i}\frac{dz}{z - a_i} - \mathbf{P}_{\widetilde{l}_j}\frac{dz}{z - a_j},$$

where $\mathbf{P}_*$ are the projectors on the appropriate $\mathrm{Res}\nabla$-invariant subspaces; it is the $gl(2)$-connection. In order to get $sl(2)$-connection we have to add the suitable 1-form

$$\widetilde{\nabla}'' = \nabla'' + \frac{1}{2}\left(\mathbf{1}_2\frac{dz}{z - a_j} - \mathbf{1}_2\frac{dz}{z - a_i}\right),$$

where $\mathbf{1}_2$ denotes the identity 2×2 matrix.

For two points $a_i, a_j \in S$ consider the modified $SL(2)$-bundle

$$\mathcal{L}'' = (a_j, l_j^+)^{\mathrm{up}} \circ (a_i, l_i^-)^{\mathrm{low}}\mathcal{L}$$

with modified logarithmic connection ∇'' defined above. This provides a nontrivial trans-formations of the coarse moduli space $\mathcal{M}_n$ of rank 2 bundles with fixed horizontal iso-morphism ϕ and logarithmic connection with fixed eigenvalues of residues on $\mathbb{P}^1$; in other words we have the Hecke correspondence on $\mathcal{M}_n$ as follows.

Proposition. ([O]) The modified pair $(\mathcal{L}'', \widetilde{\nabla''})$ is an element of the coarse moduli space $\mathcal{M}_n$. The eigenvalues of $\mathrm{Res}_a\widetilde{\nabla''}$, $a \in S$ are

$$\{\lambda_1, \ldots, \lambda_i + \frac{1}{2}, \ldots, \lambda_j - \frac{1}{2}, \ldots, \lambda_n\}$$

for the case of a pair of modifications at distinct points $a_i, a_j \in S$; for a pair of modifica-tions at one point $a_k \in S$, the eigenvalues are

$$\{\lambda_1, \ldots, \lambda_k + 1, \ldots, \lambda_n\}.$$

In this way, we have birational isomorphisms between the moduli spaces with different parameters, or between different initial data spaces; the group structure is isomorphic to the affine Weyl group $\mathfrak{W}(\widehat{C_n})$. For precise description of the discrete symmetries of our system and their action on the local solutions see [O].

3   Separation of variables

In this section, following [AL], we describe our initial data and construct étale coordinates on the open subset of $\mathcal{M}_n(2)$. It appears that these coordinates are separated coordinated in the sense of Sklyanin. Originally the recipe for the separation of variables was intro-duced in [FMcL] for the periodic Toda model. Then this procedure was generalized to the case of the Gaudin model by Sklyanin ([Skl]). Our calculation of separated variables in terms of $\Omega(\mathfrak{M})$-valued operator $L(z)$ coincides with Sklyanin's "magic recipe". In this way we give a geometric interpretation of the Sklyanin's separation of variables for the Gaudin model.

We generalize the results of the Arinkin and Lysenko work [AL] and present the cal-culations for an arbitrary number n of singularities; however, we use the ideas from [AL], in particular, two linear-algebraic lemmas.

Fix a collection $\lambda_1, \ldots, \lambda_n$ of complex numbers and the modulus $\mathfrak{M}$ with the support S at distinct points $a_1, ..., a_n$ on $\mathbb{P}^1$. The group of projective automorphisms of the Riemann sphere being three-dimensional, it is natural to restrict ourselves to the case of $n \geq 3$. Suppose $\mathcal{L}$ be a rank 2 bundle on $\mathbb{P}^1$ equipped with a fixed horizontal isomorphism ϕ : $\mathrm{Det}\mathcal{L} \simeq \mathcal{O}$ and a connection ∇ with singularities at $\mathfrak{M} = \sum a_i$; the eigenvalues of $\mathrm{Res}_{a_i}\nabla$ are $(\lambda_i, -\lambda_i)$, $i = 1, \ldots, n$.

3.1   Stable bundles

Let us discuss the definition of *stability* of our data. We consider the moduli space of vector bundles of rank 2 and we permanently control the pair $(\mathcal{L}, \nabla)$ to be indecomposable in order to provide the stability. For these purposes we put the following eigenvalue condition

$$\sum \epsilon_i \lambda_i \notin \mathbb{Z}, \qquad (\epsilon_1, \ldots, \epsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n,$$

which guarantees the irreducibility of the pair "bundle $\mathcal{L}$ with the connection ∇" and implies the stability of this pair. Indeed, given a ∇-invariant rank 1 sub-bundle $\mathcal{L}_1 \subset \mathcal{L}$ equipped with a connection $\nabla_1 := \nabla|_{\mathcal{L}_1}$ then $(\mathcal{L}_1)|_{a_i} \subset \mathcal{L}|_{a_i}$ is an eigen-space of $\mathrm{Res}_{a_i}\nabla$ and $\mathrm{Res}_{a_i}\nabla_1$ is an eigenvalue of $\mathrm{Res}_{a_i}\nabla$. In this way we get $\mathrm{Res}_{a_i}\nabla_1 = \pm\lambda_i$ but from the other hand $\sum \mathrm{Res}_{a_i}\nabla_1 = -\deg\mathcal{L}_1 \in \mathbb{Z}$ contradicts our eigenvalue-condition.

Moreover, our bundle $\mathcal{L}$ with the trivial determinant is in general nontrivial and may have a structure $\mathcal{O}(k) \oplus \mathcal{O}(-k)$. The value of k depends on n and it is defined by the stability of the construction in the following way. Let $\mathcal{L}_0 := \mathcal{O}(k)$ be a sub-bundle then by irreducibility we have a non-zero map

$$\nabla_0 : \mathcal{L}_0 \to (\mathcal{L}/\mathcal{L}_0) \otimes \Omega(\mathfrak{M})$$

which implies

$$\deg\mathcal{L}_0 \leq \deg(\mathcal{L}/\mathcal{L}_0) + \deg\Omega(\mathfrak{M}) = 0 - \deg\mathcal{L}_0 + n - 2, \quad \text{hence,} \quad k \leq \frac{n-2}{2}.$$

We consider the moduli space of pairs $(\mathcal{L}, \nabla)$ and look after the automorphism group of the pair. We demand $Aut(\mathcal{L}, \nabla) = \mathbb{C}^*$ and we assume that there are no ∇-invariant sub-bundles $\mathcal{L}_0 \subset \mathcal{L}$.

3.2 The map $(\mathcal{L}, \nabla) \mapsto (\mathcal{L}_0 \subset \mathcal{L}, \nabla)$

We shall act in the following way. Suppose that we can choose a *distinguished* sub-bundle $\mathcal{L}_0 \subset \mathcal{L}$. Then we will investigate the features of a (semi)stable element $(\mathcal{L}, \nabla) \in \mathcal{M}_n(2)$ looking at its restriction on the (non-invariant) *distinguished* sub-bundle. We have seen that for $(\mathcal{L}, \nabla) \in \mathcal{M}_n$ the structure of our bundle $\mathcal{L}$ can be $\mathcal{O}(k) \oplus \mathcal{O}(-k)$ for some k but, for example, if $k = 0$ and $\mathcal{L} \simeq \mathcal{O} \oplus \mathcal{O}$ then there is no way to choose the distinguished sub-bundle. The fact is that a bundle of an odd degree always has a distinguished sub-bundle, and it is in this way that we have to modify our bundle.

Take a point from S, say, a_1 and consider the bundle $\widetilde{\mathcal{L}} := (a_1, l_1^+)^{\text{low}}\mathcal{L}$. The natural embedding $\widetilde{\mathcal{L}} \subset \mathcal{L}$ provides an isomorphism $\mathcal{M}_n(2) \simeq \mathcal{M}'_n(2)$ with the moduli space of the following collections.

$$(\widetilde{\mathcal{L}}, \widetilde{\nabla} := \nabla|_{\widetilde{\mathcal{L}}}; \widetilde{\phi} : \mathrm{Det}\widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1); (\lambda_1^+, \lambda_1^-), \ldots, (\lambda_n^+, -\lambda_n^-)).$$

Here $\widetilde{\mathcal{L}}$ is a rank 2 bundle on $\mathbb{P}^1$ with a fixed horizontal isomorphism $\widetilde{\phi} : det\mathcal{L} \simeq \mathcal{O}(-a_1)$ and with a logarithmic connection $\widetilde{\nabla}$ with singularities at $\{a_1, \ldots, a_n\}$. The eigenvalues of $Res_{a_i}\widetilde{\nabla}$ are $(\lambda_1^+, \lambda_1^-) := (\lambda_1, 1 - \lambda_1)$ at a_1 and $(\lambda_i^+, \lambda_i^-) := (\lambda_i, -\lambda_i)$ at a_i, $i = 2, \ldots, n$. The dimension of the vector space of embeddings $\mathcal{L}/\mathcal{L}_0 \simeq \mathcal{O}(-k) \hookrightarrow \mathcal{L}$ for $k > 0$ equals

$$\dim \mathrm{Hom}(\mathcal{O}(-k), \mathcal{O}(k)) = 2k + 1 = 3, \ldots, 2 \cdot \left[\frac{n-2}{2}\right] + 1.$$

Thus, we can choose a sub-bundle $\mathcal{O}(-k)$ passing through at least $2k + 1$ of n lines $l_i^+ := \ker(\mathrm{Res}_{x_i} - \lambda_i)$ and then at least one line lies neither in $\mathcal{L}_0$, nor in our chosen $\mathcal{O}(-k)$, as we assume the bundle $(\mathcal{L}; \phi; l_i, i = 1, \ldots, n)$ to be irreducible. Thus we get the distinguished sub-bundle $\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}}$ with possible values of degree $\deg\widetilde{\mathcal{L}_0} := k' = 0, \ldots, [\frac{n-2}{2}]$.

For example, in both cases $n = 4$ and $n = 5$ the structure of $\mathcal{L}$ can be only $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(1) \oplus \mathcal{O}(-1)$; nevertheless for $n = 4$ the modified bundle is always $\widetilde{\mathcal{L}} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ and for $n = 5$ it can be either $\mathcal{O} \oplus \mathcal{O}(-1)$, or $\mathcal{O}(1) \oplus \mathcal{O}(-2)$, since the direction of the modification l_1^+ can lie in $\mathcal{L}_0 \simeq \mathcal{O}(1)$.

3.3 $\mathcal{M}'_n(2)$ as a moduli space of FH-sheaves

The algebraic variety $\mathcal{M}_n \simeq \mathcal{M}'_n$ is non-compact and consists of locally closed strata $\mathcal{M}^{k'}$, which can be interpreted as the moduli space of the following collections.

$$(\mathcal{O}(k') \subset \widetilde{\mathcal{L}}; \widetilde{\nabla}; \widetilde{\phi} : \mathrm{Det}\widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1); (\lambda_1^+, \lambda_1^-), \ldots, (\lambda_n^+, \lambda_n^-))$$

indexed by k'. The maximal value of k' depends on the parity of n: if n is even, then $k' = \frac{n-4}{2}$, and if n is odd, then $k' = \frac{n-3}{2}$.

Pick a collection of points $y_1, \ldots, y_{k'} \in \mathbb{P}^1$, and fix an isomorphism $\widetilde{\mathcal{L}_0} \simeq \mathcal{O}(y_1 + \ldots + y_{k'})$; then choose a connection ∇_0 with respect to this isomorphism with k' simple poles precisely at $y_1, \ldots, y_{k'}$ such that

$$\nabla_0 : \widetilde{\mathcal{L}_0} \longrightarrow \widetilde{\mathcal{L}_0} \otimes \Omega(y_1 + \ldots + y_{k'}), \qquad \mathrm{Res}_{y_i} \nabla_0 = 1.$$

Fixing the connection ∇_0 we get a distinguished trivialization (section) $\mathcal{O} \hookrightarrow \widetilde{\mathcal{L}_0}$ of our sub-bundle.

Restrict the connection on the sub-bundle $\widetilde{\mathcal{L}_0}$ and consider the map

$$B := \widetilde{\nabla}|_{\widetilde{\mathcal{L}_0}} - \nabla_0 : \quad \widetilde{\mathcal{L}_0} \to \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M}).$$

In this way we obtain the maps

$$f_{k'} : \mathcal{M}^{k'} \to M_1 := \text{ moduli space of } (\widetilde{\mathcal{L}_0} \simeq \mathcal{O}(k') \subset \widetilde{\mathcal{L}}, B),$$

where $\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}_0} \simeq \mathcal{O}(-k' - 1)$ and $B : \mathcal{T}(-\mathfrak{M}) \hookrightarrow \widetilde{\mathcal{L}}$ for $\mathcal{T}(-\mathfrak{M}) := \Omega(\mathfrak{M})^{-1}$.

Using the maps $f_{k'}$ we construct the maps from our moduli space $\mathcal{M}'_n$ to the moduli space of the so-called Drinfeld FH-sheaves (see [D2]):

$$\{\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}} \mid \widetilde{\mathcal{L}}/(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})) \simeq \Delta_{n-3}\},$$

where $\dim\Gamma(\mathbb{P}^1, \Delta_{n-3}) = n - 3$ and $supp(\Delta_{n-3})$ is in codimension one.

To present the strata of $\mathcal{M}'_n(2)$ as moduli spaces we have to reconstruct the element $(\widetilde{\mathcal{L}}, \widetilde{\nabla}) \in \mathcal{M}'_n(2)$ from the FH-sheaf $A = (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}})$.

Proposition. [AL] Let A be an FH-sheaf of level $n - 3$ and let R_i be an operator $\widetilde{\mathcal{L}}|_{a_i} \to \widetilde{\mathcal{L}}|_{a_i}$ with eigenvalues $\lambda_i^{\pm}$. Then, on the stratum $\mathcal{M}^0$ there is a unique connection $\widetilde{\nabla}$ such that in the above notations
(i) $\widetilde{\nabla}|_{\widetilde{\mathcal{L}_0}} = d + B$ for the unique connection $d : \widetilde{\mathcal{L}_0} \to \widetilde{\mathcal{L}_0} \otimes \Omega$ the unique connection;
(ii) $Res_{a_i} \nabla = R_i$;
(iii) $(\widetilde{\mathcal{L}}, \widetilde{\nabla}) \in \mathcal{M}'_n(2)$.

In this way we identify the generic stratum $\mathcal{M}^0$ with the moduli space of certain FH-sheaves. On the other strata the connection $\widetilde{\nabla}$ is not unique and in the following two subsections we prove the analogous proposition for all the strata. In the next subsections we give a simple construction from linear algebra and calculate the affine space of connections $\widetilde{\nabla}$ that satisfy conditions (i)-(iii).

3.4 A construction from the linear algebra

In terms of the linear algebra our description of stable pairs $(\widetilde{\mathcal{L}}, \widetilde{\nabla})$ is nothing but a reconstruction of the operator $L(z)$ such that $(\widetilde{\mathcal{L}}, \partial_z - L(z)) \in \widetilde{\mathcal{M}'_n}$ from the first row B of the operator L and the eigenvalues of the residues. Let $V_0 \subset V \simeq \mathbb{C}^2$ be a complete flag of vector spaces and let $R_0 \in \mathrm{Hom}(V_0, V)$.

Lemma A. [AL] Let $\lambda^+ \neq \lambda^- \in \mathbb{C}$ and put $\mathbf{R} := \{R \in \mathrm{End}(V)$ such that $R|_{V_0} = R_0$ and the eigenvalues of R are $\lambda^+, \lambda^-\}$, $\mathbf{L} := \{(l^+ \neq l^-) | l^{\pm} \subset V, \dim l^{\pm} = 1$ with $(R_0 - \lambda^{\mp})(V_0) \subset l^{\pm}\}$. Then the map

$$F : \mathbf{R} \longrightarrow \mathbf{L}, \quad R \mapsto (\ker(R - \lambda^+) = \mathrm{im}(R - \lambda^-), \ker(R - \lambda^-))$$

is bijective.

Proof. Clearly, F is injective, so let us check the surjectivity. For $(l^+, l^-) \in \mathbf{L}$ denote the corresponding projectors by $P_{\pm} : V \to V/l^{\pm} \simeq l^{\mp}$; one has $P_+ + P_- = \mathrm{Id}$. The condition $(R_0 - \lambda^{\mp})(V_0) \subset l^{\pm}$ implies $P^{\mp}(R_0 - \lambda^{\mp})(V_0) = 0$, or, $P^-(R_0 - \lambda^-)(V_0) + P^+(R_0 - \lambda^+)(V_0) = 0$; hence, $R_0 = (\lambda^+ P^+ + \lambda^- P^-)|_{V_0}$ and for $R := (\lambda^+ P^+ + \lambda^- P^-) \in \mathbf{R}$ we have $F(R) = (l^+, l^-)$. $\blacksquare$

One can make the similar calculations for the case $l^+ = l^-$ and proof the analogous statement.

Lemma B. Let $\lambda := \lambda^+ = \lambda^- \in \mathbb{C}$ and put $\mathbf{R} := \{R \in \mathrm{End}(V)$ such that $R|_{V_0} = R_0$ and R has the only one eigenvalue $\lambda\}$, $\mathbf{L} := \{(l \neq l') | l, l' \subset V, \dim l, l' = 1$ with $(R_0 - \lambda)(V_0) \subset l$ and $(R_0 - \lambda)(l') \subset V_0\}$. Then the map

$$F : \mathbf{R} \longrightarrow \mathbf{L}, \quad R \mapsto (\ker(R - \lambda), \mathrm{im}(R - \lambda))$$

is bijective. $\blacksquare$

3.5 Calculation of the affine space of connections

In this subsection we use the notations and technique from [SGA5]. Let us remark that the connection $\widetilde{\nabla}$ that satisfies conditions (i)-(iii) exist locally on $\mathbb{P}^1$. Given an open subset $U \subset \mathbb{P}^1$, denote by $\mathcal{C}(U)$ the set of all local connections $\widetilde{\nabla} = \nabla_0 - L(z)$ on U. Given $\widetilde{\nabla}, \widetilde{\nabla}' \in \mathcal{C}(U)$, then $E := \widetilde{\nabla} - \widetilde{\nabla}'$ is an element of $H^0(U, \underline{Hom}(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}} \otimes \Omega)) \simeq \underline{Hom}(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}_0}, \widetilde{\mathcal{L}_0} \otimes \Omega)$ such that $E|_{\widetilde{\mathcal{L}_0}} = 0$ and $\mathrm{Tr} E = 0$. Denote by $\mathcal{E}(U)$ the set of such local homomorphisms. Clearly, $\mathcal{C}$ is an $\mathcal{E}$-torsor and the obstruction to the existence of a global connection lies in $H^1(\mathbb{P}^1, \mathcal{E}(\mathfrak{M}))$ which by the Serre duality is dual to

$$H^0(\mathbb{P}^1, \{E \in \mathrm{End}(\mathcal{L}) \,|\, \mathrm{Tr}\, E = 0, \, E(a_i)(l_i^+) \subset l_i^+ \,\}) = \{0\}.$$

In this way a global connection always exists, but it is not unique.

Thus we parameterize the space of connections by the matrix element $L(z)_{21}$ of $L(z)$, and as we have seen

$$L(z)_{21} \in \underline{Hom}(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}_0}, \widetilde{\mathcal{L}_0} \otimes \Omega) \simeq \mathcal{E}.$$

Let us calculate the space of connections on each stratum $\mathcal{M}^{k'}$, assuming that the FH-sheaf $A = (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}})$ is generic.

On the stratum $\mathfrak{M}^0$ we have the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) & \xrightarrow{\ A\ } & \widetilde{\mathcal{L}} & \longrightarrow & \bigoplus_{i=1}^{n-3} \delta_{x_i} \otimes p_i \otimes \mathcal{T}_{x_i} & \longrightarrow & 0 \\
& & \| & & \| & & & & \\
0 & \longrightarrow & \mathcal{O} \oplus \mathcal{O}(2-n) & \xrightarrow{\ A\ } & \mathcal{O} \oplus \mathcal{O}(-1) & & & &
\end{array}
$$

For all x_i we have $\operatorname{im} A(x_i) \nsubseteq \widetilde{\mathcal{L}_0} \simeq \mathcal{O}$, hence, all $p_i < \infty$ and the map

$$
\mathfrak{M}^0 \longrightarrow \underbrace{K'_n \times \ldots \times K'_n}_{n-3}
$$

is an isomorphism at a generic point (modulo the assumption that all x_i are distinct). The sheaf $\mathcal{E} \simeq \underline{Hom}(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}_0}, \widetilde{\mathcal{L}_0} \otimes \Omega)$ is of degree -1, hence, any $\mathcal{E}$-torsor is trivial and we have the unique connection recovered by our procedure.

On the stratum $\mathfrak{M}^1$ we have

$$
A := \operatorname{Id} \oplus B : \quad \mathcal{O} \oplus (\mathcal{T}(-\mathfrak{M})) \longrightarrow \widetilde{\mathcal{L}} \simeq \mathcal{O}(y_1) \oplus \mathcal{O}(-2)
$$

and, if $x_i = y_1$ for some i, then we make the upper modification at x_i in the infinite direction, and $p_i = \infty$. Note that the case $p_i = \infty$ corresponds to the point at infinity of $\overline{K'_n} := \mathbb{P}(\mathcal{O} \oplus \Omega(\mathfrak{M}))$ and it means that the modification in $(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}))|_{x_i}$ is performed in the direction of $\mathcal{O}|_{x_i} \subset (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}))|_{x_i}$. In this way we have a map

$$
\mathfrak{M}^1 \longrightarrow \overline{K'_n} \times \underbrace{K'_n \times \ldots \times K'_n}_{n-4}
$$

The sheaf $\mathcal{E} = \underline{Hom}(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}_0}, \widetilde{\mathcal{L}_0} \otimes \Omega)$ is isomorphic to $\underline{Hom}(\mathcal{O}(-2), \mathcal{O}(1) \otimes \Omega) \simeq \mathcal{O}(1)$ and on this stratum the affine space of connections is 2-dimensional.

On the stratum $\mathfrak{M}^{k'}$ we have

$$
A := \operatorname{Id} \oplus B : \quad \mathcal{O} \oplus (\mathcal{T}(-\mathfrak{M})) \longrightarrow \widetilde{\mathcal{L}} \simeq \mathcal{O}(y_1 + \ldots + y_{k'}) \oplus \mathcal{O}(-k'-1),
$$

hence,

$$
\mathfrak{M}^{k'} \longrightarrow \underbrace{\overline{K'_n} \times \ldots \times \overline{K'_n}}_{k'} \times \underbrace{\overline{K'_n} \times \ldots \times \overline{K'_n}}_{n-3-k'}.
$$

Besides, $\mathcal{E} \simeq \underline{Hom}(\mathcal{O}(-k'-1), \mathcal{O}(k') \otimes \Omega) \simeq \mathcal{O}(2k'-1)$, and on this stratum the affine space of connections is parameterized by $L(z)_{21}$, and it is $2k'$-dimensional.

3.6 Étale coordinates on $\mathcal{M}'_n(2)$ at the generic point

Recall that from

$$
L|_{\widetilde{\mathcal{L}_0}} = B : \quad \mathcal{T}(-\mathfrak{M}) \hookrightarrow \widetilde{\mathcal{L}}
$$

and $\operatorname{Id} : \mathcal{O} \hookrightarrow \widetilde{\mathcal{L}}$ we have constructed FH-sheaf

$$
A := \operatorname{Id} \oplus B : \quad \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \longrightarrow \widetilde{\mathcal{L}}.
$$

Moreover, in the generic situation we have the following factorization

$$A = A_1 \circ \ldots \circ A_{n-3}, \quad A_i = (x_i, p_i)^{\mathrm{up}}, \ i = 1, \ldots, n-3$$

which implies $\mathrm{Det}\, A(x_i) = 0$, $i = 1, \ldots, n-3$; hence, in the neighborhood of a point x_i we have

$$A(x_i) = \begin{pmatrix} B_{11} & B_{12} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_{11}(x_i) = p_i, \quad B_{12}(x_i) = 0, \ i = 1, \ldots, n-3.$$

By Lemmas A and B we recover the operator $L(z)$ from the following data; $L(z)|_{\mathcal{L}_0} = A(z)$, $\mathrm{Res}_{a_i} L(z)$ has the eigenvalues λ_i^+, λ_i^- and the trace $\mathrm{Tr} L(z) = (z - a_1)^{-1}$.

The $n-3$ zeroes of B_{12} are exactly the x_i, $i = 1, \ldots, n-3$ étale coordinates on $\mathcal{M}_n'$. One readily identify this calculation with the analogous one from [Skl].

In this way we are given an exact sequence

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \xrightarrow{A} \widetilde{\mathcal{L}} \longrightarrow \delta_{x_i} \otimes p_i \otimes \mathcal{T}_{x_i} \longrightarrow 0, \qquad i = 1, \ldots, n-3,$$

where $A_1 \circ \ldots \circ A_{n-3} = A : \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \to \widetilde{\mathcal{L}}$ is a composition of the upper modifications $(x_i, p_i)^{\mathrm{up}}$. The directions of the modifications $p_i \subset (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}))|_{x_i}$ are one-dimensional subspaces and they are parameterized by the surface $\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$. So, we would like to construct maps $\mathcal{M}_n'(2) \longrightarrow \mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$ and parameterize $\mathcal{M}_n'(2)$ by $\{x_i, p_i\}$, $i = 1, \ldots, n-3$. In fact $\{x_i, p_i\}$, $i = 1, \ldots, n-3$ are étale coordinates on an open subset of $\mathcal{M}_n'(2)$.

There is no ordering on our array of A_i, $i = 1, \ldots, n-3$ and we have the action of the symmetric group $\mathfrak{S}_{n-3}$ on our construction of $\mathcal{M}_n'$; a change of order of the upper modifications $A_i = (x_i, p_i)^{\mathrm{up}}$, $i = 1, \ldots, n-3$ induces a nontrivial automorphism of $\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))^{n-3}$. In this way, there is no a map from $\mathcal{M}_n'$ to $\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))^{n-3}$, but there is one to the quotient

$$\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))^{(n-3)} := \underbrace{\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M})) \times \ldots \times \mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))}_{n-3} / \mathfrak{S}_{n-3}.$$

One may also consider the $(n-3)!$-branched covering $\widetilde{\mathcal{M}_n'}$ of $\mathcal{M}_n'$, and study the interplay between $\widetilde{\mathcal{M}_n'}$ and $\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))^{n-3}$.

3.7 Description of the fibers $F_i = \Omega(\mathfrak{M})|_{a_i}$

Let us analyze the behavior of the map A when x_i tends to $a \in S$. At a singular point a we have two conditions foon the eigen-values of the residue $L_a := \mathrm{Res}_a \nabla$:

$$\mathrm{Tr}\, L_a = 0 \quad \text{and} \quad \mathrm{Det}\, L_a = \lambda_a^+ \cdot \lambda_a^-, \quad a \in S.$$

We reconstruct the operator

$$L(z)|_{x_i \to a} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & -L_{11} \end{pmatrix}$$

and obtain

$$L_{11} = B_{11} = p_i dz, \quad \mathrm{Res}L_{12} \to 0, \quad \mathrm{Res}L_{21} = \frac{\mathrm{Det}\, L_a - p_i^2}{\mathrm{Res}L_{12}}.$$

We see that $\mathrm{Res}\, L(z)_{21}$ can have a finite value only when $p_i \to \lambda_a^{\pm}$ and we have to calculate the limit of L_{21} by the L'Hospital rule considering the next terms of expansions of $\mathrm{Det}\, L_a - p_i^2$ and $\mathrm{Res}L_{12}$. From the geometric point of view we just make a blow-up (a σ-process) at this point.

Consider $K_n := \mathrm{Tot}(\mathbb{P}^1, \mathcal{O} \oplus \Omega(\mathfrak{M}))$ with the fibers $F_a \subset K_n$ at $a \in \mathbb{P}^1$. Since $\mathrm{Res}_a : \Omega(\mathfrak{M})|_a \xrightarrow{\sim} \mathbb{C}$, we have $R_a : F_a \xrightarrow{\sim} \mathbb{C}$; blow up K_n at $2n$ points $R_a^{-1}(\lambda_a^{\pm})$ and get

$$K_n' := (\mathrm{Bl}_{R_a^{-1}(\lambda_a^{\pm})} K_n) \setminus \bigsqcup \widetilde{F_a},$$

where $\widetilde{F_a}$ are the pre-images of the fibers $F_a \subset K_n$ after the blow-up processes. Finally, we have a map

$$\widetilde{\mathcal{M}_n'} \longrightarrow \underbrace{K_n' \times \ldots \times K_n'}_{n-3}.$$

For $n = 4$ this map is an isomorphism but, in general as we have seen in 3.5 this map is neither injective nor surjective; nevertheless, it is an isomorphism at the generic point of $\widetilde{\mathcal{M}_n'}$.

4 Compactification and dynamics of the system

We have found the étale coordinates $\{x_i, p_i\}$, $i = 1, \ldots, n-3$ on the open subset of the initial data space $\mathcal{M}_n'(2)$ and now we investigate a compactification of $\mathcal{M}_n'(2)$ in terms of these variables. On the open subset of the moduli space $\mathcal{M}_n(2)$ is isomorphic to the symmetric power of the surface K_n'; each factor is $(K_n')_{(i)} \simeq \mathrm{Bl}_{\lambda_i^{\pm}} \mathbb{P}(\mathcal{O} \oplus \Omega(\mathfrak{M})) \setminus \Theta_{(i)}$, $i = 1, \ldots, n-3$. In the same way the factors of the compactifying divisor D are the components

$$(\Theta_{(i)})^{\mathrm{red}} = s_\infty + \widetilde{F_1} + \ldots + \widetilde{F_n} \subset \mathcal{B}l_{\lambda_i^{\pm}} \mathbb{P}(\mathcal{O} \oplus \Omega(\mathfrak{M})),$$

where s_∞ is the infinite section $\mathbb{P}(\mathcal{O} \oplus \Omega^1(\mathfrak{M})) \setminus \mathit{Tot}(\Omega^1(\mathfrak{M})$ and $\widetilde{F_i}$ are the pre-images of the fibres $F_i := \Omega^1(\mathfrak{M})|_{a_i} \subset \mathrm{Tot}(\Omega^1(\mathfrak{M}))$ at singular points $a_1, \ldots, a_n$. In this way the compactifying divisor is

$$D = \boldsymbol{\Theta}_n + \sum_{r=1}^{n-3} (\Theta_{(i)})^r \times (K_n')^{n-3-r},$$

where $\boldsymbol{\Theta}_n = D \cdot D$ is the complete self-intersection cycle and evidently $\boldsymbol{\Theta}_n = \Theta^{(n-3)}$.

In this section we present a natural compactification of $\mathcal{M}_n'(2)$ due to Drinfeld (see [D2]). Namely, we use the interpretation of $\mathcal{M}_n'(2)$ as moduli space of FH-sheaves with ceratin restricting conditions. Thus to complete such moduli space one just has to remove the restricting conditions on FH-sheaves. Moreover, this construction gives a description of the compactifying set as a moduli space of certain FH-sheaves. At the end of the section in 4.3 we give a geometric presentation of isomonodromic dynamics in terms of $\boldsymbol{\Theta}_n$.

4.1 Drinfeld's compactification

Note that all the moduli spaces considered here are the coarse moduli spaces, and we do not discuss here the interplay between the corresponding algebraic stacks. Recall the interpretation of the moduli space $\mathcal{M}_n(2)$ in terms of certain FH-sheaves step by step.

First, we present an isomorphism $\mathcal{M}_n(2) \xrightarrow{\sim} \mathcal{M}'_n(2)$, where $\mathcal{M}'_n(2)$ is the moduli space of rank 2 bundles $\widetilde{\mathcal{L}}$ with the horizontal isomorphism $\widetilde{\phi} : \mathrm{Det}\widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1)$. This bundle is equipped with a logarithmic connection $\widetilde{\nabla}$ with fixed eigenvalues $\{\lambda_i^+, \lambda_i^-\}$ of the residues $\mathrm{Res}_{a_i}\nabla$. This isomorphism is given by the lower modification $\widetilde{\mathcal{L}} := (a_1, l_1^+)^{\mathrm{low}}$ in the direction $l_1^+ := \ker(\mathrm{Res}_{a_1}\nabla - \lambda_1) \subset \widetilde{\mathcal{L}}|_{a_1}$ and the eigenvalues of the residues of the connection are

$$\lambda_i^+ = \lambda_i,\ i = 1, \ldots, n, \qquad \lambda_1^- = 1 - \lambda_1, \quad \lambda_i^- = -\lambda_i,\ i > 1.$$

The upper modification $(a_1, l_1^-)^{\mathrm{up}}$ defines the inverse isomorphism.

Second, the pair $(\widetilde{\mathcal{L}}, \nabla)$ is irreducible and contains a distinguished sub-sheaf $\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}}$ of degree $k' = 0, \ldots, \left[\frac{n-3}{2}\right]$. We fix a set of distinct points $y_1, \ldots, y_{k'} \in \mathbb{P}^1$ such that

$$\widetilde{\mathcal{L}_0} \xrightarrow{\sim} \mathcal{O}(y_1 + \ldots + y_{k'})$$

and consider a connection

$$\nabla_0 : \widetilde{\mathcal{L}_0} \longrightarrow \widetilde{\mathcal{L}_0} \otimes \Omega(y_1 + \ldots + y_{k'});$$

fixing ∇_0 we define a distinguished section $\mathcal{O} \subseteq \widetilde{\mathcal{L}_0}$.

Denote by M_1 the coarse moduli space of triples

$$(\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}},\ A,\ \widetilde{\phi}),$$

where

$$\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}_0} \simeq \mathcal{O}(-k'-1), \quad k' = 0, \ldots, \left[\frac{n-3}{2}\right],$$

and $A \in \mathrm{Hom}(\widetilde{\mathcal{L}_0}, \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M}))$ such that $\mathrm{im}(A) \nsubseteq \widetilde{\mathcal{L}_0} \otimes \Omega(\mathfrak{M})$. There is a map $\mathcal{M}'_n(2) \to M_1$, defined by

$$(\widetilde{\mathcal{L}}, \nabla, \widetilde{\phi}) \mapsto (\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}},\ A := \nabla|_{\widetilde{\mathcal{L}_0}} - \nabla_0,\ \widetilde{\phi}).$$

Note that on the open subset the moduli space M_1 is isomorphic to the $(n-3)$-th symmetric power of the non-compact surface $\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$ and the condition $\mathrm{im}(A) \subset \widetilde{\mathcal{L}_0} \otimes \Omega(\mathfrak{M})$ defines the infinite section $s_\infty \subset \mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$.

Third, identify the moduli space $\mathcal{M}'_n(2)$ with the coarse moduli space of the following collections;

$$(\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}},\ A,\ \widetilde{\phi};\ l_1^+, l_1^-, \ldots, l_n^+, l_n^-),$$

such that
(i) $(\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}},\ A,\ \widetilde{\phi})$ is a point of the moduli space M_1;
(ii) $l_i^\pm \subset \widetilde{\mathcal{L}}|_{a_i}$ is the one-dimensional subspace defined by

$$(\mathrm{Res}_{a_i} A - \lambda^\mp)(\widetilde{\mathcal{L}_0}|_{a_i}) \subset l_i^\pm;$$

(iii) $l_i^+ \neq l_i^-$.

In the previous section it was shown that on the open subset we may identify the $(n-3)$!-covering $\widetilde{\mathcal{M}_n(2)}$ with the $(n-3)$-th power of the surface K_n'. The surface $K_n' \simeq \mathrm{Bl}_{\lambda^\pm}\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$ is obtained by blowing up $K_n = \mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$ at $2n$ points $(a_i, \lambda^\pm)$.

Denote by M_2 the coarse moduli space of $(\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}}, A, \widetilde{\phi}; l_1^+, l_1^-, \ldots, l_n^+, l_n^-)$ such that only the conditions (i), (ii) are satisfied, and (iii) is hold for all a_i except for some $a \in S$. It is the condition (iii) that defines the union of pre-images of the fibers $F_i := \Omega^1(\mathfrak{M})|_{a_i} \subset \mathrm{Tot}(\Omega^1(\mathfrak{M}))$ and the infinite section s_∞. Thus M_2 is a divisor on $\mathcal{M}_n'(2)$; moreover, it naturally completes our moduli space $\mathcal{M}_n'(2)$ and we identify M_2 with the compactifying divisor D. It is the Drinfeld compactification in the sense of [D2].

Denote by M_2' the coarse moduli space of $(\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}}, A, \widetilde{\phi}; l_1^+, l_1^-, \ldots, l_n^+, l_n^-)$ such that only the conditions (i), (ii) are satisfied; the condition (iii) does not hold for all $a_i \in S$. Identify M_2' with the complete self-intersection locus of the compactifying divisor D and denote it by $\boldsymbol{\Theta}_n$.

4.2 D and $\boldsymbol{\Theta}_n$ in terms of FH-sheaves

As we have seen the divisor D (and its complete self-intersection $\boldsymbol{\Theta}_n$) may be identified with the coarse moduli space of $(\widetilde{\mathcal{L}_0} \subset \widetilde{\mathcal{L}}, A, \widetilde{\phi})$ with $A \in \mathrm{Hom}(\widetilde{\mathcal{L}_0}, \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M}))$, satisfied the following two conditions:

(1) $\mathrm{im}(A) \subset \widetilde{\mathcal{L}_0} \otimes \Omega(\mathfrak{M})$;

(2) $l_a^- := (\mathrm{Res}_a A - \lambda^+)(\widetilde{\mathcal{L}_0}|_a) = l_i^+ := (\mathrm{Res}_a A - \lambda^-)(\widetilde{\mathcal{L}_0}|_a)$ for some (and for all) $a \in S$.

Condition (2) implies $l_a^+ = l_a^- = (\widetilde{\mathcal{L}_0}|_a)$, and for $a = a_i$ it defines the fibre F_i. Altogether, conditions (2) imply (1), and the (1) means that all the subspaces l_i^+ and l_i^-, for $i = 1, \ldots, n$, coincide with $\widetilde{\mathcal{L}_0}|_{a_i}$ and define the (blow-up of the) intersection of all fibers F_i, $i = 1, \ldots, n$. In this way the conditions (1) and (2) give us components

$$\Theta_{(i)} := (s_\infty + s_\infty + F_1 + \ldots + F_n) \subset \overline{K_n}.$$

Consider the Hecke correspondence between our moduli space $\boldsymbol{\Theta}_n$ of FH-sheaves $(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}})$ of level $n-3$ and the moduli space $\boldsymbol{\Theta}_n'$ of FH-sheaves $(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}}')$ of level zero. In other words, let us perform $n-3$ lower modifications of our bundle $\widetilde{\mathcal{L}}$ of degree -1 at distinct points $a \in \{a_1, \ldots, a_n\}$ in the direction $l_a^+ = l_a^-$. Thus, after such procedure we get the bundle $\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})$ of degree $2-n$ for the chosen directions $l_a^+ = l_a^-$ lie in $\widetilde{\mathcal{L}_0}|_{a_i}$.

It is more convenient to investigate the complete self-intersection locus $\boldsymbol{\Theta}_n$ of the compactifying divisor $D = \overline{\mathcal{M}_n'(2)} \setminus \mathcal{M}_n'(2)$. In fact, it is isomorphic to the coarse moduli space of collections

$$(\widetilde{\mathcal{L}_{\Theta_n}}, \nabla_{\Theta_n}, \phi'),$$

with the fixed eigenvalues of residues of the connection. Here $\widetilde{\mathcal{L}_{\Theta_n}}$ is a bundle of degree $2-n$ on $\mathbb{P}^1$ with the horizontal isomorphism $\phi' : \mathrm{Det}\widetilde{\mathcal{L}}' \xrightarrow{\sim} \mathcal{O}(-a_1 - a_{n-2})$, and the connection ∇_{Θ_n} has the following eigenvalues of the residues. For $a_i = a_1, \ldots, a_{n-2}$ the residues $\mathrm{Res}\nabla_{\Theta_n}$ have eigenvalues $(\lambda_i, 1 - \lambda_i)$ and for $a_i = a_{n-1}, a_n$ the eigenvalues are $(\lambda_i, -\lambda_i)$.

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The connection ∇_{Θ_n} exists but it is not unique. Let us calculate the dimension of the appropriate affine space. Given an open subset $U \subset \mathbb{P}^1$, denote by $\mathcal{C}(U)$ the set of all local connections $\nabla_{\Theta_n} = \nabla_0 - L(z)$ on U. For two connections $\nabla'_{\Theta_n}, \nabla''_{\Theta_n} \in \mathcal{C}(U)$ their difference $E' := \nabla''_{\Theta_n} - \nabla'_{\Theta_n}$ is an element of $H^0(U, \underline{Hom}(\widetilde{\mathcal{L}_{\Theta_n}}, \widetilde{\mathcal{L}_{\Theta_n}} \otimes \Omega)) \simeq \underline{Hom}(\widetilde{\mathcal{L}_{\Theta_n}}/\mathcal{O}, \mathcal{O} \otimes \Omega)$, such that $E'|_{\mathcal{O}} = 0$ and $\mathrm{Tr}\, E' = 0$. Let $\mathcal{E}_{\Theta_n}(U)$ be the set of such morphisms E'. Then $\mathcal{C}$ has a natural structure of $\mathcal{E}_{\Theta_n}$-torsor and the obstruction to the existence of a global connection lies in the group $H^1(\mathbb{P}^1, \mathcal{E}_{\Theta_n}(\mathfrak{M}))$, which is dual to $H^0(\{E' \in \mathrm{End}(\mathcal{L}_{\Theta_n}) | \mathrm{Tr}\, E' = 0, E'(a_i)(l_i^+) \subset l_i^+\}) = \{0\}$ by the Serre duality. We define our global connection by reconstructing the row $(L(z)_{21}, -L(z)_{11})$ of the operator $L(z)$ and the connection is parameterized by the element $L(z)_{21}$ that lies in $\underline{Hom}(\widetilde{\mathcal{L}_{\Theta_n}}/\mathcal{O}, \mathcal{O} \otimes \Omega) \simeq \mathcal{E}_{\Theta_n}$. In this way

$$\mathcal{E}_{\Theta_n} \simeq \Omega_{\mathbb{P}^1}^{\otimes 2}(\mathfrak{M}) \simeq \mathcal{O}(n-4)$$

and the dimension of the affine space of the connection ∇_{Θ_n} on the bundle $\widetilde{\mathcal{L}_{\Theta_n}} \simeq \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})$ equals $n-3$.

At last, just note that one can interpret the divisor D as a moduli space of certain FH-sheaves of level zero considering the appropriate Hecke correspondence.

4.3   Dynamics of the $sl(2)$ isomonodromic system

In the final part of the section let us present the étale coordinates $\{x_i, p_i\}$, $i = 1, \ldots, n-3$ as parameters of the Hecke correspondence between the coarse moduli spaces $\boldsymbol{\Theta}_n$ and $\mathcal{M}'_n(2)$, and interpret them in terms of the apparent singularities of the connection ∇. Precisely, consider the space of sections of the sheaf $\mathcal{E}_{\Theta_n} \simeq \underline{Hom}(\widetilde{\mathcal{L}_{\Theta_n}}/\mathcal{O}, \mathcal{O} \otimes \Omega_{\mathbb{P}^1}^1)$ on the moduli space $\boldsymbol{\Theta}'_n$ of the collections

$$(\widetilde{\mathcal{L}_{\Theta_n}}, \nabla_{\Theta_n}; \phi' : \mathrm{Det}\, \widetilde{\mathcal{L}_{\Theta_n}} \xrightarrow{\sim} \mathcal{O}(-a_1 - \ldots - a_{n-2}); (\widetilde{\lambda_i^+}, \widetilde{\lambda_i^-}), i = 1, \ldots, n)$$

for $\widetilde{\lambda_i^+} := \lambda_i$ and $\widetilde{\lambda_i^-} = 1 - \lambda_i$, for $a_i \neq a_{n-1}, a_n$; the rest $\widetilde{\lambda_i^-} = -\lambda_i$ for $a_i = a_{n-1}, a_n$. Note here that the configuration $(\widetilde{\mathcal{L}_{\Theta_n}}; l_1^+, \ldots, l_n^+)$ is semi-stable in our notation, since we have

$$\mathrm{Aut}(\widetilde{\mathcal{L}_{\Theta_n}}) \simeq \begin{matrix} End(\mathcal{O}) & \oplus & \mathrm{Hom}(\mathcal{T}(-\mathfrak{M}), \mathcal{O}) \\ \oplus & & \oplus \\ \mathrm{Hom}(\mathcal{O}, \mathcal{T}(-\mathfrak{M})) & \oplus & \mathrm{End}(\mathcal{T}(-\mathfrak{M})) \end{matrix} \simeq \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(n-2),$$

hence, $\mathrm{Aut}(\widetilde{\mathcal{L}_{\Theta_n}}; l_1^+, \ldots, l_n^+) \simeq \mathbb{C}^*$.

As we have seen, the space of sections of the sheaf $\mathcal{E}_{\Theta_n}$ on $\boldsymbol{\Theta}_n$ has dimension $(n-3)$; hence,

$$\dim \Gamma(\boldsymbol{\Theta}_n, \mathcal{E}_{\Theta_n}) + \dim \boldsymbol{\Theta}_n = 2 \cdot (n-3),$$

that is, exactly the dimension of the moduli space $\mathcal{M}'_n(2)$. Take a collection of distinct points $\{x_1, \ldots, x_{n-3}\} \subset \mathbb{P}^1$ and a collection of one-dimensional subspaces $p_i \subset \widetilde{\mathcal{L}_{\Theta_n}}|_{x_i}$, $i = 1, \ldots, n-3$ and perform the modifications

$$A := (x_1, p_1)^{\mathrm{up}} \circ \ldots \circ (x_{n-3}, p_{n-3})^{\mathrm{up}} : \quad \widetilde{\mathcal{L}_{\Theta_n}} \longrightarrow \widetilde{\mathcal{L}},$$

where $\widetilde{\mathcal{L}}$ is a rank 2 bundle of degree -1 on $\mathbb{P}^1$. As it was shown, this gives us a map from $\mathcal{M}'_n(2)$ to the symmetric product $(\text{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M})))^{(n-3)}$ at the generic point.

Next, choose the unique connection $\nabla_{\Theta_n}(p_1, \ldots, p_{n-3}) \in \mathcal{E}_{\Theta_n}$ such that the subspaces $p_1, \ldots, p_{n-3}$ are invariant for it. The modification of the connection is

$$A : \quad \nabla_{\Theta_n}(p_1, \ldots, p_{n-3}) \longrightarrow \widetilde{\nabla} = \nabla_{\Theta_n}(p_1, \ldots, p_{n-3}) - \sum_{i=1}^{n-3} \mathbf{P}_{p_i} \frac{dz}{z - x_i},$$

where $\mathbf{P}_{p_i}$ are the projectors on the (invariant) one-dimensional subspaces $p_1, \ldots, p_{n-3}$. Note that this correspondence is isomonodromic and the terms $\mathbf{P}_{p_i} \dfrac{dz}{z - x_i}$ does not change the monodromy of the connection and the points $x_1, \ldots, x_{n-3}$ are called *apparent singularities* of the connection $\widetilde{\nabla}$. Originally, the apparent singularities were introduced in [F] by L. Fuchs; more detailed approach to the Fuchsian equations and systems one can find in the books [B], and [AB].

In this way, we interpret the Hecke correspondence between the moduli spaces $\mathbf{\Theta}_n$ and $\mathcal{M}'_n(2)$ as the deformation of the most degenerate locus $\mathbf{\Theta}_n$ of D in the fibred space $\text{Tot}(\mathbf{\Theta}_n, \mathcal{E}_{\Theta_n})$ performed by modifications of the connection ∇_{Θ_n} with apparent singularities $\mathbf{P}_{p_i} \dfrac{dz}{z - x_i}$. In the case when $x_i \in S$, the dynamics of the isomonodromic system becomes discrete and presented by the lattice C_n; for calculations see the proposition in Section 2; for applications to the relations between the special functions, – solutions of the Fuchsian equations, – see the paper [O].

5   An example: the Painlevé-VI system

Now, we illustrate our constructions of the étale coordinates on the initial data space and its compactification in the simplest example of the $sl(2)$-isomonodromic system with four marked points called the sixth Painlevé system. In this section we suppose that $\mathcal{L}$ is a rank 2 vector bundle on $\mathbb{P}^1$ with $\text{Det}\mathcal{L} \simeq \mathcal{O}$ and a logarithmic connection ∇ with eigenvalues $(\lambda_i, -\lambda_i)$ of the residues at four singularities a_i, $i = 1, .., 4$. So we have a modulus $\mathfrak{M} = \sum a_i$ and modulo projective transformations of $\mathbb{P}^1$ by the three-dimensional group $PGL(2, \mathbb{C})$ we can suppose $\mathfrak{M} = 0 + 1 + t + \infty$, where $t := r(a_1, a_2, a_3, a_4)$ is the cross-ratio; however, $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1(\mathfrak{M})$.

Following the ideas of previous sections, we shall investigate the geometry of the moduli space $\mathcal{M}_4$ of such pairs $(\mathcal{L}, \nabla)$; its biggest cell is isomorphic to the symplectic quotient $\mathcal{O}_1 \times \ldots \times \mathcal{O}_4 // SL(2, \mathbb{C})$. We identify it with the phase space of the Schlesinger system with four points on $\mathbb{P}^1$, called the sixth Painlevé equation. We define suitable coordinates using the geometric construction of the Schlesinger system from [AL]. Then we construct a natural compactification of the phase space also considered in [AL], which is coincide with the Okamoto compactification constructed in [Oka]. At the end, we discuss the geometric realization of the dynamics and the interplay with the apparent singularities which is original.

First, consider the configuration space of the Painlevé-VI system. It is the moduli space of so-called quasi-parabolic bundles $\mathcal{N}_4$. Precisely, $\mathcal{N}_4$ is the moduli space of the collections

$$(\mathcal{L}; \quad \phi : \text{Det}\mathcal{L} \simeq \mathcal{O}; \quad l_1, \ldots, l_4),$$

where $\mathcal{L}$ is a rank 2 bundle with a horizontal isomorphism ϕ and $l_i \subset \mathcal{L}|_{a_i}$ are one-dimensional subspaces; there is a canonical surjection $\pi : \mathcal{M}_4 \twoheadrightarrow \mathcal{N}_4$ defined by

$$(\mathcal{L}, \nabla; \lambda_1, \ldots, \lambda_n) \mapsto (\mathcal{L}; l_i^+ := \ker(\mathrm{Res}_{a_i} \nabla - \lambda_i), \ i = 1, \ldots, 4).$$

In fact, the configuration space $\mathcal{N}_4(2)$ is parameterized by the x coordinate. As we have seen each pair x_i, p_i naturally parameterize the non-trivial bundle $\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$; in this way it is interesting to calculate the map π.

5.1 Geometry of $\mathcal{N}_4(2)$

Let us describe the configuration space of four eigenvectors in the two-dimensional vector space or the configurations of four points l_1, l_2, l_3, l_4 in $\mathbb{P}^1$ modulo the action of $PGL(2)$. In our description we follow Mumford's approach (see [MS]).

The invariant of the configuration is the cross-ratio

$$r(l_1, l_2, l_3, l_4) := \frac{l_1 - l_3}{l_1 - l_4} \cdot \frac{l_2 - l_4}{l_2 - l_3};$$

naturally, it is a coordinate on $\mathcal{N}_4(2)$. Since we have the action of the projective group $PGL(2, \mathbb{C})$ we can suppose

$$l_1 = X, \quad l_2 = 1, \quad l_3 = 0, \quad l_4 = \infty, \quad \text{hence,} \quad r(l_1, l_2, l_3, l_4) = X;$$

let us calculate the behavior of $X = r(l_1, l_2, l_3, l_4)$ under the action of the permutational factor-group

$$0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow \mathfrak{S}_4 \longrightarrow \mathfrak{S}_3 \longrightarrow 1.$$

The possible values of the cross-ratio are $1 - X$, $X^{-1}, 1 - X^{-1}$. For example the value

$$1 - X = 1 - \frac{l_1 - l_3}{l_1 - l_4} \cdot \frac{l_2 - l_4}{l_2 - l_3} = \frac{l_4 - l_3}{l_4 - l_1} \cdot \frac{l_2 - l_1}{l_2 - l_3}$$

corresponds to two different permutations: $(14) := l_1 \leftrightarrow l_4$ and $(23) := l_2 \leftrightarrow l_3$. Thus, it corresponds to two different quasi-parabolic bundles: one with $\{l_4 = l_1 \neq l_2 \neq l_3 \neq l_1\}$ and another with $\{l_3 = l_2 \neq l_1 \neq l_4 \neq l_2\}$. In this way if the two of the four points on the Riemann sphere try to glue, then two others glue too: $X \to \infty$ if and only if $1 \to 0$. Moreover, for each value $X = 0$, $X = 1$, $X = \infty$, there are two different configurations of quasi-parabolic bundles. Note that the configuration of the quasi-parabolic bundle for the value $X = r(l_1, l_2, l_3, l_4) = t = r(a_1, a_2, a_3, a_4)$ corresponds to the nontrivial bundle $\mathcal{L} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Choose a basis in the two-dimensional fiber of our bundle: $\mathcal{L}|_{a_i} := \langle l_2, l_3 \rangle$; then

$$\begin{cases} l_1 = \alpha \cdot l_2 + \beta \cdot l_3 = l_2 + l_3; \\ l_2 = 1 \cdot l_2 + 0 \cdot l_3; \\ l_3 = 0 \cdot l_2 + 1 \cdot l_3; \\ l_4 = \gamma \cdot l_2 + \delta \cdot l_3 = l_2 + r(\alpha, \beta, \gamma, \delta) \cdot l_3 \end{cases}, \qquad X = r(\alpha, \beta, \gamma, \delta);$$

consider the action of pairs of modifications on our bundle (see Section 2):

$$(a_2, l_2)^{up} : \mathcal{L} \to \mathcal{L}', \quad \langle l_2, l_3 \rangle \to \langle l_2' := \frac{l_2}{X - a_2}, l_3 \rangle,$$

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$$(a_3, l_3)^{low} : \mathcal{L}' \to \widetilde{\mathcal{L}}, \quad \langle l_2', l_3 \rangle \to \langle \widetilde{l_2} := \frac{X - a_3}{X - a_2} \cdot l_2, l_3 \rangle.$$

We have the modified eigenvectors

$$\begin{cases} \widetilde{l_1} = \left(\dfrac{X - a_3}{x - a_2} \cdot l_2 + l_3 \right)_{X=a_1} = l_2 + l_3; \\[2mm] l_2 = 1 \cdot l_2 + 0 \cdot l_3; \\[2mm] l_3 = 0 \cdot l_2 + 1 \cdot l_3; \\[2mm] \widetilde{l_4} = \left(\dfrac{X - a_3}{X - a_2} \cdot l_2 + r(\alpha, \beta, \gamma, \delta) \cdot l_3 \right)_{X=a_4} = r(a_1, a_2, a_3, a_4) \cdot l_2 + r(\alpha, \beta, \gamma, \delta) \cdot l_3 \end{cases} ;$$

if $r(\alpha, \beta, \gamma, \delta) \to t = r(a_1, a_2, a_3, a_4)$, then $\widetilde{l_4} \to \widetilde{l_1}$. An analogous calculation with the pair of modifications $(a_1, l_1)^{up}(a_4, l_4)^{low}$ shows that the case $\widetilde{l_2} \to \widetilde{l_3}$ gives the same value $x = t$, hence, this value corresponds to two different nontrivial quasi-parabolic bundles, and finally we have the following

Statement. ([AL]) $\mathcal{N}_4$ is isomorphic to two copies of $\mathbb{P}^1$ glued outside $\{0, 1, t, \infty\}$. The action of the pairs of modifications on $\mathcal{N}_4$ is evident and it presents the affine $\widehat{D}_4$ lattice.

5.2 Geometry of $\mathcal{M}_4(2)$

Describe the geometry of the moduli space of the collections

$$(\mathcal{L}, \nabla; \phi : \mathrm{Det}\mathcal{L} \simeq \mathcal{O}; \lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

where $\mathcal{L}$ is a rank 2 vector bundle with fixed holomorphic structure ϕ on the determinant, and ∇ is a logarithmic connection with fixed eigenvalues of the residues at the points of the support S of the modulus $\mathfrak{M} = 0 + 1 + t + \infty$ on $\mathbb{P}^1$. Put the eigenvalue condition

$$\sum \epsilon_i \lambda_i \notin \mathbb{Z}, \qquad (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in (\mathbb{Z}/2\mathbb{Z})^4$$

which provides the irreducibility of our pair $(\mathcal{L}, \nabla)$. Our notion of stability (see 3.1) of our pair $(\mathcal{L}, \nabla)$ implies that neither of the eigenvectors $l_i^+ := \ker(\mathrm{Res}_{x_i} \nabla - \lambda_i)$ may lie in the sub-bundle $\mathcal{L}_0 \simeq \mathcal{O}(1)$. Modify our bundle, say, at $(\infty, l_\infty^+)^{low}$, we necessarily get the bundle $\widetilde{\mathcal{L}} \simeq \mathcal{O} \oplus \mathcal{O}(-\infty)$; this modification presents an isomorphism of $\mathcal{M}_4$ with $\mathcal{M}_4'$, which is the moduli space of the following collections.

$$(\widetilde{\mathcal{L}}, \widetilde{\nabla}; \quad \widetilde{\phi} : \mathrm{Det}\widetilde{\mathcal{L}} \simeq \mathcal{O}(-\infty); \quad (\lambda_1, -\lambda_1), \dots, (\lambda_\infty, 1 - \lambda_\infty)).$$

In this way, we get a uniquely defined sub-bundle

$$\widetilde{\mathcal{L}} \supset \widetilde{\mathcal{L}_0} \simeq \mathcal{O}$$

with the standard connection d. Restrict our connection to the sub-bundle and consider the operator

$$A(z) := \mathrm{Id} \oplus (\nabla|_{\widetilde{\mathcal{L}_0}} - \partial_z) : \quad \mathcal{O} \oplus \widetilde{\mathcal{L}_0} \longrightarrow \widetilde{\mathcal{L}} \otimes \Omega^1(\mathfrak{M}).$$

Our pair is irreducible, $\mathrm{Im}(\nabla|_{\widetilde{\mathcal{L}_0}} - \partial_z)(\widetilde{\mathcal{L}_0}) \not\subseteq \widetilde{\mathcal{L}_0}$, hence,

$$A(z) := \mathrm{Id} \oplus (\nabla|_{\widetilde{\mathcal{L}_0}} - \partial_z) : \quad \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \longrightarrow \widetilde{\mathcal{L}}.$$

The determinant $\mathrm{Det} A(z)$ has a simple pole at some point x and, moreover, $A(z) = (x, p)^{\mathrm{up}}$; the variables x and p are the canonical coordinates on the two-dimensional initial data space $\mathcal{M}_4$ of our isomonodromic system. The surface $\mathcal{M}_4$ is noncompact and has a structure of a fibred space over $\mathcal{N}_4$. Note that in our case the cohomological calculations are very simple: $\mathcal{E} \simeq \mathcal{O}(-2)^* \otimes \mathcal{O}(-1) \otimes \Omega \simeq \mathcal{O}(-1)$ and $H^1(\mathcal{E}) = 0$, hence, $\mathcal{M}_4 \simeq K'_4$

5.3 Geometry of the Painlevé-VI system

As we have seen the moduli space $\mathcal{M}'_4(2)$ is the non-compact surface
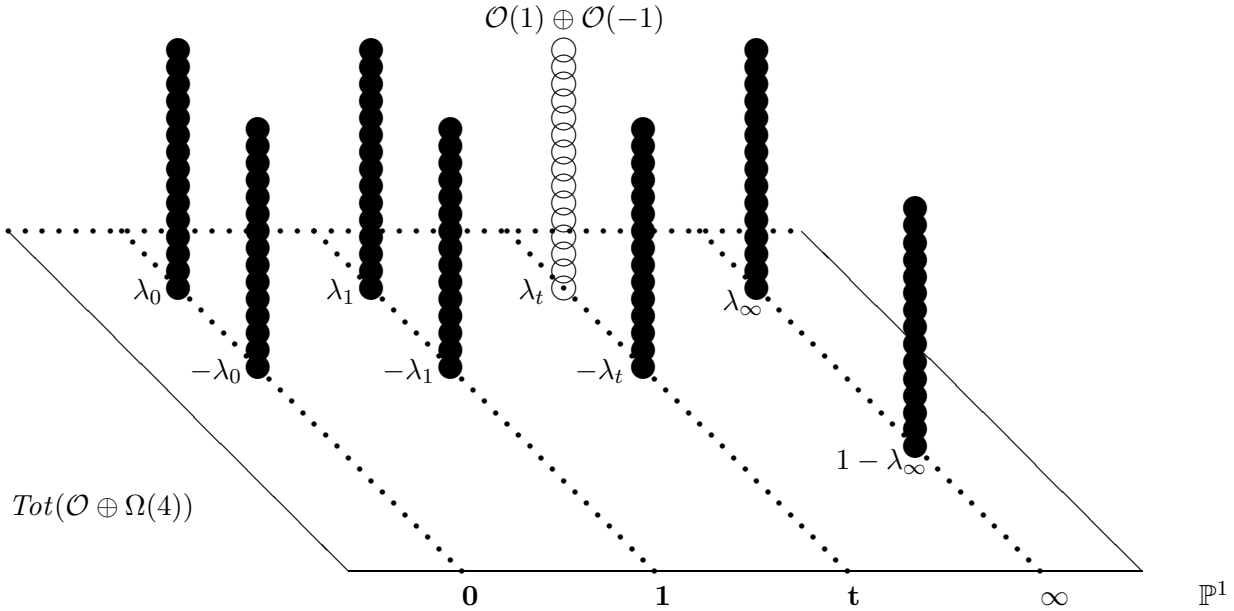
The exceptional divisor at a point (t, λ_t) corresponds to the collection $(\widetilde{\mathcal{L}}, \widetilde{\nabla}; \quad \widetilde{\phi} : \mathrm{Det}\widetilde{\mathcal{L}} \simeq \mathcal{O}(-\infty); \quad (\lambda_1, -\lambda_1), \ldots, (\lambda_\infty, 1 - \lambda_\infty))$ with a nontrivial bundle $\widetilde{\mathcal{L}} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

In this way we have the following presentation of the initial data space

$$\mathcal{M}_4(2) \simeq K'_4 := (\mathrm{Bl}_{R^{-1}(\lambda_i^\pm)}\mathrm{Tot}(\mathbb{P}^1, \mathcal{O}(2))) \setminus \bigsqcup \widetilde{F}_i, \quad i = 1, \ldots; 4$$

it is isomorphic to the moduli space of the stable FH-sheaves

$$(\mathcal{O} \oplus \mathcal{T}(-0 - 1 - t - \infty) \subset \mathcal{O} \oplus \mathcal{O}(-\infty))$$

of level 1. In other words, the coordinates (x, p) on the initial data space present it as the moduli space of exact sequences
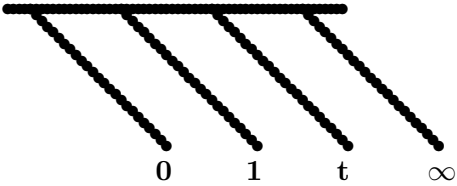
$$0 \longrightarrow \mathcal{O} \oplus \mathcal{T}(-4) \longrightarrow \mathcal{O} \oplus \mathcal{O}(-\infty) \longrightarrow \delta_x \otimes p \otimes \mathcal{T}_x \longrightarrow 0$$

such that $p < \infty$ and if $x = a \in S$ then $p = \lambda_a^{\pm}$.

Consider the natural symplectic form $\varpi = dx \wedge dp$ on $\mathbb{P}(\mathcal{O} \oplus \Omega(4))$, and let us look at its behavior when $x \in S$. At singular points of the connection the dynamics is discrete and performed by the lattice $\widehat{F}_4$. We blow-up eight points $(x, p) = (a, \lambda_a^{\pm})$, $a \in S$, on the surface $\mathbb{P}(\mathcal{O} \oplus \Omega(4))$; locally this procedure performed by $p = s \cdot x$ for s a coordinate on the exceptional divisor. Then, remove four fibers $\widetilde{F_a} := \{a, p\} \subset \mathbb{P}(\mathcal{O} \oplus \Omega(4))$ and in this way at $x = a$ we have two exceptional curves with

$$ds = \frac{dp}{x} - s \cdot \frac{dx}{x}.$$

The compactifying set is exactly the divisor of poles of the symplectic form $\varpi = dx \wedge dp$, and it performs the degeneration of an elliptic curve C. The divisor is

$$D = (2 \cdot s_{\infty} + \widetilde{F_0} + \widetilde{F_1} + \widetilde{F_t} + \widetilde{F_{\infty}})^{\mathrm{red}} =$$

$$0 \qquad 1 \qquad t \qquad \infty$$

it is defined by the conditions $p = \infty$ and $l_a^+ = l_a^-$, $a = 0, 1, t, \infty$. Let $\widetilde{\mathcal{L}}$ be the bundle corresponding to a point on the compactifying divisor and perform the lower modification, say, at $a = 0$ in the direction

$$l_0^+ \subset \mathcal{O}|_{z=0} \subset (\mathcal{O} \oplus \mathcal{O}(-1))|_{z=0}.$$

We get the bundle $\widetilde{\mathcal{L}_D} \simeq \mathcal{O} \oplus \mathcal{T}(-4)$, and we have an isomorphism of D with the moduli space of the collections

$$(\widetilde{\mathcal{L}_D}, \nabla_D, \phi', (\widetilde{\lambda_i^+}, \widetilde{\lambda_i^-})),$$

where $\widetilde{\mathcal{L}_D}$ is a bundle of degree -2 on $\mathbb{P}^1$ with the horizontal isomorphism $\phi' : \mathrm{Det}\widetilde{\mathcal{L}'} \widetilde{\rightarrow} \mathcal{O}(-0-\infty)$ and the connection ∇_D with the following eigenvalues of residues $(\widetilde{\lambda_0^+}, \widetilde{\lambda_0^-}) = (\lambda_0, 1 - \lambda_0)$,

$$(\widetilde{\lambda_1^+}, \widetilde{\lambda_1^-}) = (\lambda_1, -\lambda_1), \quad (\widetilde{\lambda_t^+}, \widetilde{\lambda_t^-}) = (\lambda_t, -\lambda_t), \quad (\widetilde{\lambda_{\infty}^+}, \widetilde{\lambda_{\infty}^-}) = (\lambda_{\infty}, 1 - \lambda_{\infty}).$$

Finally we have the following diagram

$$\mathcal{O} \oplus \mathcal{T}(-4) \quad \overset{(x,p)^{\mathrm{up}}}{\longrightarrow} \quad \mathcal{O} \oplus \mathcal{O}(-1) \quad \begin{matrix} \longrightarrow \\ \longleftarrow \end{matrix} \quad \begin{bmatrix} \mathcal{O} \oplus \mathcal{O} \\ \mathcal{O}(1) \oplus \mathcal{O}(-1) \end{bmatrix} .$$

The right two arrows $\overset{\rightarrow}{\leftarrow}$ denote the action of discrete $\widehat{F}_4$-symmetries (see [AL], [O]) and the left arrow $\overset{(x,p)^{\mathrm{up}}}{\longrightarrow}$ in terms of the connections is

$$(x, p)^{\mathrm{up}} : \widetilde{\nabla} = \nabla_D(p) - \mathbf{P}_p \frac{dz}{z - x}.$$

Note here that the connection ∇_D is not uniquely defined. Such connections on the bundle $\mathcal{O} \oplus \mathcal{O}(-2)$ form a one-dimensional affine space and we choose uniquely the connection $\nabla_D(p)$ for which the direction p is proper; otherwise, as it was shown we can get the quadratic pole of $\widetilde{\nabla}$ at $z = x$.

The term $\mathbf{P}_p \dfrac{dz}{z - x}$ does not change the monodromy of connections and the simple pole at $z = x$ is an apparent singular point for the appropriate Fuchsian system. In this way we perform the isomonodromic system Painlevé-VI as the deformation of the moduli space D by the Hecke correspondence $(x, p)^{\mathrm{up}}$.

For the interpretation of the Painlevé-VI system as a deformation of the compactifying divisor in terms of the Kodaira-Spencer theory see [T].

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